MTH 402 THEORY OF PARTIAL DIFFERENTIAL EQUATIONS 3 Units
Theory and solutions of first-order and second order linear equations. Classification, characteristics, canonical forms, Cauchy problems. Elliptic equations; Laplace's and Poisson's formulas, properties of harmonic functions. Hyperbolic equations; wave equations, retarded potential; transmission line equation, Riemann method. Parabolic equation. Diffusion equation, singularity function, boundary and initial-value problem.

1.0 Introduction

When partial derivatives are required in the mathematical formulation of some physical phenomenon, the resulting equation is called partial differential equation. A partial differential equation (PDE) is one involving the partial derivatives of one or more dependent variables with respect to two or more independent variables. The variables involved may be time and/or one or more spatial coordinates. It is convenient to indicate partial derivatives by writing independent variables as subscripts. Thus, we can write:

$$u_x \text{ for } \frac{\partial u}{\partial x}, \quad u_{xx} \text{ for } \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} \text{ for } \frac{\partial^2 u}{\partial x \partial y}$$

and so on. It is generally assumed that dependent variable, u satisfies conditions so that:

$$u_{xy} = u_{yx}$$

Examples:

Two Dimensions: If $u = u(x, y)$ is a function of two variables, the following expressions are examples of PDE;

$$\frac{\partial u}{\partial x} = u + x + y \quad \text{or} \quad u_x = u + x + y$$

$$\frac{\partial u}{\partial x} = 0 \text{ or } u_x = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \text{ or } u_x + u_y = 0, \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1 \text{ or } (u_x)^2 + \left(u_y\right)^2 = 1$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \text{ or } u_{xy} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} + u_{yy} = 0, \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} - u_{yy} =$$

Three Dimensions: If $u = u(x, y, z)$ is a function of three variables, the following expressions are PDE:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \text{ or } u_x + u_y + u_z = 0,$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1 \text{ or } (u_x)^2 + \left(u_y\right)^2 + (u_z)^2 = 1$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ or } u_{xx} + u_{yy} + u_{zz} = 0, \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0 \text{ or } u_{xx} - u_{yy} - u_{zz} = 1$$

and

$$u_{xxx} + 3y u_{xx} - x u_x u_{yy} + u\left(u_y\right)^3 = e^{xyz}$$

In these examples, (x, y) represents a point in the plane, and (x, y, z) represents a point in space. Sometimes solutions u of PDE depend also on the variable t that denotes time.

1.1 Classification of PDEs

PDEs are classified as to order, linearity, and homogeneity in much the same way as ODE's:

Order of PDE: The order of a PDE is the order of the highest derivative in the equation. For example

$$u_x = 3x^2 + 7y^5$$

Is a first-order PDE, while

$$a(u_{xx})^3 + b(u_y)^5 = c$$

is of second-order PDE.

Degree of a PDE: The degree of a PDE is the degree of the highest derivatives in the equation. The equation above,

$$a(u_{xx})^3 + b(u_y)^5 = c$$

is of degree 3.

Linear PDE: A linear PDE is one in which the dependent variable and all its partial derivatives are of first degree and each coefficient depends only on the independent variables. For example

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u$$

is a first order PDE in two independent variables x and y. The equation is homogeneous if $c(x,y) = 0$.

Quasi-Linear PDE: A quasi-linear PDE is one in which the partial derivatives of the dependent variable are of first degree, although the dependent variable itself may not be. The most general quasi-linear first-order equation in two independent variables can be put in the form:

$$au_{xx} + bu_{xy} + cu_{yy} + c(x, y, u, u_x, u_y) = 0$$

where a, b, c are functions of x and y, and c is a function of the indicated quantities.

Non-Linear PDE: A non-linear PDE is one in which at least one derivative is of degree greater than one, or there is a product of derivatives. For example,

$$a(x,y)(u_x)^2 + b(x,y)u_y = 0$$

is a non-linear equation.

1.2 Solutions of PDE's

Any function which when substituted in a PDE reduces it to an identity is said to be a solution of the equation. Given an ODE, the kind of functions that are admissible as solutions are fixed; whereas in a PDE, the functions are left arbitrary. For example, the ODE of a second order:

$$u''(x) + k^2u(x) = 0$$

has a general solution of the form:

$$u(x) = A\cos kx + B\sin kx$$

where A and B are arbitrary constants. The functions $\cos kx$ and $\sin kx$ are fixed. To obtain a definite solution, we must specify two numbers A and B.

But by the general solution of PDE, we mean a function u that has all partial derivatives occurring in the PDE that, when substituted into the equation, reduces it to an identity for all independent variables. For example, let us consider the simple first-order PDE given above:

$$u_x = 3x^2 + 7y^5 \qquad (1)$$

where x and y are independent variables and u is the unknown. Holding y fixed and integrating (1) with respect to x, we find:

$$u(x, y) = x^3 + 7xy^5 + F(y) \qquad (2)$$

where F is any differentiable function of y. Thus, it follows that:

$$u(x, y) = x^3 + 7xy^5 + e^y,$$

$$u(x, y) = x^3 + 7xy^5 - \sin y,$$

$$u(x, y) = x^3 + 7xy^5 + 2y^{\frac{3}{2}},$$

and so on, are all solution functions of (1). Similarly, it is easy to verify that:

$$u(x, y) = x + 2y,$$

$$u(x, y) = \cos(x + 2y),$$

$$u(x, y) = 3e^x e^{2y} = 3e^{x+2y},$$

each satisfy the equation:

$$2u_x - u_y = 0 \qquad (3)$$

Thus, a general solution of a PDE is a collection of all the solutions of the equation. For instance, (2) is a general solution of (1) and

$$u(x, y) = G(x + 2y) \qquad (4)$$

is a general solution of (3), where G is any differentiable function of $x + 2y$. Here, we find one of the most fundamental differences between the general solution of an ODE and that of a PDE – the general solution of an ODE contains arbitrary constants, whereas that of a PDE involves arbitrary functions.

In practice, one is seeking a particular solution of the PDE that satisfies certain auxiliary conditions arising out of a given physical situation. Most of the time, it is impossible, or at least difficult to find the needed particular solution by specialising the general solution of the PDE as is done for ODEs. This is so because it is very difficult to specialise arbitrary functions except in special situations. Therefore, it is usually best in solving PDEs to find a suitable set of particular solutions, each of which satisfies some of the prescribed auxiliary conditions, and then combine these particular solutions in some fashion so that the resulting solution function satisfies all the prescribed conditions.

1.3 Properly-posed Problems

The problems consisting of solving a PDE subject to certain conditions in the form of boundary and/or initial conditions are called initial-boundary value problems, or more simply, boundary value problems. In the study of these problems, three basic questions arise of chief importance:

a) Does a solution exist?
b) Is the solution unique?
c) Is the solution stable?

Questions concerning existence and uniqueness are standard in any study of DEs. The third question above deals with the problem of whether the solution depends continuously upon the prescribed data (both boundary and initial conditions). That is, do small changes in the prescribed data produce only small changes in the values of the solution function at each point? This question is of great concern in applications wherein the auxiliary data are determined most often by measurements and hence are only approximate, not exact as we assume in theoretical discussions. We wound certainly hope that small errors in these measurements would produce only small errors in the solution function. A boundary value problem possessing a unique, stable solution is called properly-posed or well-posed. However, it is noteworthy that problem with too many prescribed boundary and/or initial conditions is an over-specified problem and may not have a solution, and a problem that has too few prescribed conditions does not have a unique solution.

1.3.1 Types of Auxiliary Conditions

Much work has been carried out over the years to determine the types of auxiliary conditions that must be prescribed so that a given boundary value problem is properly-posed, but such an analysis here is beyond the scope of this course. We shall however discuss the boundary and initial conditions that frequently arise in the description of physical phenomena which fall mainly into four categories:

a) **Dirichlet Conditions:** In a Dirichlet boundary condition, the value of the solution at the boundary points is specified. This can be written as;

$$u(a) = \alpha \quad \text{and} \quad u(b) = \beta$$

where $u(x)$ is the solution to the differential equation, and a and b are the boundary points. The unknown function u is specified at each point on the boundary of the region of interest. For example, consider the boundary value problem for the one-dimensional heat equation:

$$u_t = ku_{xx}, \quad 0 < x < L, \qquad t > 0$$

$u(0,t) = 0, \;\; u(L,t) = 0, \qquad t > 0$

$u(x,0) = f(x), \;\; 0 < x < L$

where k is a constant, $u(x,t)$ is the temperature at position x and time t, and $f(x)$ is the initial temperature distribution. The Dirichlet boundary conditions specify that the temperature at the two ends of the rod ($x = 0$ and $x = L$) are fixed at zero. This models the situation where the rod is insulated at both ends and no heat can flow in or out.

b) **Neumann Conditions:** In a Neumann boundary condition, the derivative of the solution at the boundary points is specified. This can be written as;

$$u'(a) = \alpha \quad \text{and} \quad u'(b) = \beta$$

Values of the normal derivative of the unknown function u are prescribed at each point on the boundary of the region of interest. Consider the boundary value problem for the one-dimensional wave equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \qquad t > 0$$

$u_x(0,t) = 0, \;\; u_x(L,t) = 0, \qquad t > 0$

$u(x,0) = f(x), \qquad 0 < x < L$

$u_t(x,0) = g(x), \qquad 0 < x < L$

where c is a constant, $u(x,t)$ is the displacement of a string at position x and time t, $f(x)$ is the initial displacement, and $g(x)$ is the initial velocity. The Neumann boundary conditions specify that the endpoints of the string are fixed ($u_x(0,t) = u_x(L,t) = 0$), which models the situation where the string is attached to two fixed points.

c) **Robin Conditions:** In a Robin boundary condition, a linear combination of the value of the solution and its derivative at the boundary points is specified. This can be written as;

$$au(a) + bu'(a) = \alpha \quad \text{and} \quad cu(b) + du'(b) = \beta$$

where $a, b, c,$ and d are constants. Values of the sum of the unknown function u and its normal derivative are prescribed at each point on the boundary of the region of interest. For example, consider the boundary value problem for the two-dimensional Laplace equation:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \qquad 0 < y < H$$

$u(0, y) = 0, \; u(L, y) = 0, \; 0 < y < H$
$u_x(x, 0) + u(x, 0) = f(x), \; 0 < x < L$
$u_x(x, H) - u(x, H) = g(x), \; 0 < x < L$

where $u(x, y)$ is the potential at position $(x, y), f(x)$ and $g(x)$ are given functions, and the boundary conditions represent a physical problem in electrostatics. The Robin boundary conditions specify that the potential at the bottom boundary ($y = 0$) is a function of its gradient ($u_x(x, 0) + u(x, 0) = f(x)$), and that the potential at the top boundary ($y = H$) is a function of the difference in gradient across the boundary ($u_x(x, H) - u(x, H) = g(x)$).

d) **Cauchy Conditions:** Cauchy conditions are a type of boundary conditions for partial differential equations (PDEs) that specify both the value and the derivative of the solution on a portion of the boundary. In contrast to the Dirichlet, Neumann, and Robin boundary conditions, which typically apply to the entire boundary, Cauchy conditions apply only to a portion of the boundary. More specifically, suppose we have a PDE with independent variables x and t, and dependent variable $u(x, t)$. Then, Cauchy conditions are given in terms of the value and derivative of the solution at a fixed value of t, say $t = t_0$, on a portion of the boundary, say $x = a$:
$u(a, t_0) = f(a)$ and $u_x(a, t_0) = g(a)$
where $f(a)$ and $g(a)$ are given functions of x.

In other words, Cauchy conditions specify the initial values of the solution and its first derivative at a fixed time t_0 on a portion of the boundary. These conditions are useful in situations where the solution is known or can be easily determined at a specific time and location, and we want to propagate the solution forward or backward in time.

An example of a PDE with Cauchy conditions is the one-dimensional heat equation:

$$u_t = k u_{xx}, \quad 0 < x < L, \qquad t > 0$$

$u(x, 0) = f(x), \; 0 < x < L$
$u(a, t) = g(t), \qquad u_x(0, t) = 0, \qquad u_x(L, t) = 0, \qquad t > 0$

where k is a constant, $u(x, t)$ is the temperature at position x and time t, $f(x)$ is the initial temperature distribution, and $g(t)$ is the temperature at the boundary point $x = a$ at time t. The Cauchy conditions specify the value and derivative of the solution at $x = a$ and time t_0, which allows us to find the temperature distribution for all times $t > 0$.

2.0 Second-Order Partial Differential Equations

The most important PDEs of higher order that are encountered in mathematical physics and most engineering problems are of second order which we shall discuss below.

2.1 Classification

The general form of the second-order linear PDE in two independent variables x and y is given by:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \tag{2.1}$$

where A, B, C, E, F, G, are functions of the independent variables x, y. The equation is linear because u and its partial derivatives appear only to the first degree and the coefficients depend only on the independent variables x and y.

Second-order PDEs are classified in terms of the coefficients of the second-order terms. These PDEs are classified according to the discriminant:

$$B^2 - AC$$

as follows:

$$B^2 - AC \begin{cases} > 0 \ \Rightarrow \text{PDE is hyperbolic} \\ = 0 \ \Rightarrow \text{PDE is parabolic} \\ < 0 \ \Rightarrow \text{PDE is elliptic} \end{cases}$$

Example 2.1

1. The 1-D wave equation:

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

is an example of a *hyperbolic* PDE, where $A = 1,\ B = 0,\ C = -\frac{1}{c^2}$

2. The 1-D diffusion or heat conduction equation:

$$u_{xx} = \frac{1}{h^2} u_t$$

is an example of a *parabolic* PDE, where $A = 1,\ B = C = 0$.

3. The 2-D Laplace's equation:

$$u_{xx} + u_{yy} = 0$$

is an example of *elliptic* PDE, where $A = C = 1,\ B = 0$.

In general, we know that the coefficients A, B, C are not constants. This implies that there may exist a curve in the xy-plane along which the PDE may change from one form to another. A practical example occurs in fluid dynamics, where the transition from subsonic to supersonic flows corresponds to a change from elliptic to hyperbolic equation.

Example 2.2: In what regions of the xy-plane is the equation:

$$u_{xx} + xu_{yy} + u_y = 0$$

hyperbolic, parabolic or elliptic?

Solution:

Here, $A = 1,\ B = 0,\ C = x$. Thus, the discriminant is:

$$B^2 - AC = -x \begin{cases} > 0 \text{ if } x < 0 \text{ (hyperbolic)} \\ = 0 \text{ if } x = 0 \text{ (parabolic)} \\ < 0 \text{ if } x > 0 \text{ (elliptic)} \end{cases}$$

Thus, the PDE is hyperbolic when $x < 0$, parabolic when $x = 0$, and elliptic when $x > 0$.

2.2 Transformation

If the PDE is complicated, it may be necessary to reduce the equation to its simplest form before attempting to obtain a solution. This can be done by a transformation of the independent variables. We therefore introduce new independent variables r and s such that:

$$r = r(x,y) \quad \text{and} \quad s = s(x,y)$$

called canonical or standard coordinates; where r and s possesses continuous second partial derivatives. Then,

$u_x = u_r r_x + u_s s_x$

$u_y = u_r r_y + u_s s_y$

but

$$u_{xy} = (u_r r_x + u_s s_x)_y = u_r . r_{xy} + r_x . u_{ry} + u_s . s_{xy} + s_x . u_{sy} \qquad (2.2)$$

Now, expanding u_{ry} and u_{sy}, we have:

$u_{ry} = u_r(u_y) = u_r(u_r r_y + u_s s_y) = u_{rr} r_y + u_{rs} s_y$ and $u_{sy} = u_s(u_r r_y + u_s s_y) = u_{rs} r_y + u_{ss} s_y$

Substituting for u_{ry} and u_{sy} in (2.2), we obtain:

$$\begin{aligned} u_{xy} &= u_r r_{xy} + r_x(u_{rr} r_y + u_{rs} s_y) + u_s s_{xy} + s_x(u_{rs} r_y + u_{ss} s_y) \\ &= u_{rr} r_x r_y + u_{rs} r_x s_y + u_{rs} r_y s_x + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy} \\ &= u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy} \qquad (2.3) \end{aligned}$$

To obtain u_{xx} *and* u_{yy}, we substitute for x and y in (2.3):

$$\begin{aligned} u_{xx} &= u_{rr} r_x r_x + u_{rs}(r_x s_x + r_x s_x) + u_{ss} s_x s_x + u_r r_{xx} + u_s s_{xx} \\ &= u_{rr} {r_x}^2 + 2u_{rs} r_x s_x + u_{ss} {s_x}^2 + u_r r_{xx} + u_s s_{xx} \qquad (2.4) \end{aligned}$$

and

$$u_{yy} = u_{rr} {r_y}^2 + 2u_{rs} r_y s_y + u_{ss} {s_y}^2 + u_r r_{yy} + u_s s_{yy} \qquad (2.5)$$

Employing these in (2.1), denoted by:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y)$$

yields:

$$\begin{aligned} &A[u_{rr} {r_x}^2 + 2u_{rs} r_x s_x + u_{ss} {s_x}^2 + u_r r_{xx} + u_s s_{xx}] \\ &\qquad + 2B[u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy}] \\ &\qquad + C[u_{rr} {r_y}^2 + 2u_{rs} r_y s_y + u_{ss} {s_y}^2 + u_r r_{yy} + u_s s_{yy}] + D[u_r r_x + u_s s_x] + E[u_r r_y + u_s s_y] \\ &\qquad + Fu = G \qquad (2.6) \end{aligned}$$

Collecting the coefficients, we have:

$$A' u_{rr} + 2B' u_{rs} + C' u_{ss} + D' u_r + E' u_s + F' = G' \qquad (2.6)$$

where;

$$\begin{aligned} A' &= A{r_x}^2 + 2Br_x r_y + C{r_y}^2 \\ B' &= Ar_x s_x + 2B(r_x s_y + r_y s_x) + Cr_y s_y \\ C' &= A{s_x}^2 + 2Bs_x s_y + C{s_y}^2 \qquad (2.7) \\ D' &= Ar_{xx} + 2Br_{xy} + Cr_{yy} + Dr_x + Er_y \\ E' &= As_{xx} + 2Bs_{xy} + Cs_{yy} + Ds_x + Es_y \\ F' &= F, \quad G' = G \end{aligned}$$

But the discriminant of (2.6) is:

$$B'^2 - A'C' = (B^2 - AC)\left(r_x{}^2 s_y{}^2 - 2r_x r_y s_x s_y + r_y{}^2 s_x{}^2\right) = (B^2 - AC)(r_x s_y - r_y s_x)^2 \qquad (2.8)$$

or

$$B'^2 - A'C' = (B^2 - AC)J^2 \qquad (2.9)$$

where

$$J = \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix}$$

is the Jacobian of the transformation. Equation (2.8) implies that the form of the equation is invariant with respect to an arbitrary transformation of the coordinates, since J^2 is always positive real quantities, and the sign of $B'^2 - A'C'$ is the same as that of $B^2 - AC$.

2.2 Canonical Forms

If in the form of A' , B', C' in (2.7), we let;

$$\alpha = \frac{r_x}{r_y}, \ \ \beta = \frac{s_x}{s_y}$$

we get:

$$\begin{aligned} A' &= [A\alpha^2 + 2B\alpha + C]r_y{}^2 \\ B' &= [A\alpha\beta + B(\alpha + \beta) + C]r_y s_y \qquad (2.10) \\ C' &= [A\beta^2 + 2B\beta + C]s_y{}^2 \end{aligned}$$

We determine the functions r and s so that $A' = 0$ or $C' = 0$, to get:

$A\alpha^2 + 2B\alpha + C = 0$ or $A\beta^2 + 2B\beta + C = 0$, with roots'

$$\alpha = \frac{-B - \sqrt{B^2 - AC}}{A} = \frac{r_x}{r_y}, \ \ \beta = \frac{-B + \sqrt{B^2 - AC}}{A} = \frac{s_x}{s_y} \qquad (2.11)$$

where $A \neq 0$, so that the coordinate transformation will be non-singular.

The functions r and s may be identified with the constants c_1 and c_2 obtained by integrating $dr = 0$ and $ds = 0$, i.e.,

$r = c_1$ and $s = c_2$.

$dr = 0$ and $ds = 0$, gives $r_x dx + r_y dy = 0$ and $s_x dx + s_y dy = 0$

or

$$\frac{dy}{dx} + \frac{r_x}{r_y} = 0 \ \text{ or } \ \frac{dy}{dx} = -\frac{r_x}{r_y} \ \text{ and } \ \frac{dy}{dx} + \frac{s_x}{s_y} = 0 \ \text{ or } \ \frac{dy}{dx} = -\frac{s_x}{s_y}$$

or from (2.11),

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A} \ \text{ or } \ \frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A} \qquad (2.12)$$

Equations (2.12) gives two families of curves called characteristics if $B^2 - AC > 0$; one family of characteristic curves if $B^2 - AC = 0$; and no family of characteristics if $B^2 - AC < 0$.

a) **Hyperbolic case:** Since $B^2 - AC > 0$, it follows that the roots α and β are real. From (2.6), the canonical form in this case is:

$$u_{rs} + f(r, s, u, u_r, u_s) = 0 \qquad (2.13)$$

where r and s are obtained by integrating equations (2.12), and $A' = C' = 0$.

b) **Parabolic case:** Since $B^2 - AC = 0$, it follows that both roots α and β equall $\frac{B}{A}$. Integrating one of the equations (2.12) and identifying the result with r and s can be taken as any function which is linearly independent with r. The canonical form is obtained by setting $A' = B' = 0$ in (2.6), and we obtain:

$$u_{ss} + g(r, s, u, u_r, u_s) = 0 \qquad (2.14)$$

Note: In the parabolic case, if we identify the result of integrating $\frac{dy}{dx} = \frac{B}{A}$ with s, and taking r to be any linearly independent function with s, we will get the canonical form:

$$u_{rr} + g(r, s, u, u_r, u_s) = 0 \qquad (2.15)$$

which is obtained by setting $B' = C' = 0$ in (2.6).

c) **Elliptic case:** Since $B^2 - AC < 0$, it follows that the two roots α and β are complex conjugates:

$$\alpha = \frac{-B - i\sqrt{B^2 - AC}}{A}, \quad \beta = \frac{-B + i\sqrt{B^2 - AC}}{A} \qquad (2.16)$$

The results are formally the same as those of the hyperbolic case, except that we have complex rather than real quantities. Since it is more convenient to deal with real quantities, we make further transformation:

$$\mu = r + s, \qquad \nu = i(r - s)$$

Thus, the term u_{rs} in (2.13) from equation (2.3) may be written as:

$$u_{rs} = u_{\mu\mu}\mu_r\mu_s + u_{\mu\nu}(\mu_r\nu_s + \mu_s\nu_r) + u_{\nu\nu}\nu_r\nu_s + u_\mu\mu_{rs} + u_\nu\nu_{rs} \quad or \quad u_{rs} = u_{\mu\mu} + u_{\nu\nu};$$

Since

$$\mu_r = \mu_s = 1, \qquad \nu_r = i, \qquad \nu_s = -i, \qquad \mu_{rs} = \nu_{rs} = 0$$

Hence, the required canonical from is:

$$u_{\mu\mu} + u_{\nu\nu} + h\left(\mu, \nu, u, u_\mu, u_\nu\right) = 0 \qquad (2.17)$$

Summary

	Hyperbolic	Parabolic	Elliptic
Sign of $B^2 - AC$	Positive	Zero	Negative
Family of characteristics	2	1	0
Canonical or standard form	u_{rs}	u_{rr} or u_{ss}	$u_{rr} + u_{ss}$

Examples 2.3: Reduce to canonical form, and hence find the general solution if the equation is hyperbolic or parabolic:

a) $u_{xx} + 2u_{xy} + u_{yy} = 2$

b) $u_{xx} - 2u_{xy} + u_{yy} = 0$

c) $u_{xx} + 2u_{xy} + 2u_{yy} = 4$

d) $u_{xx} + u_{yy} = 0$

Solution:

a) $u_{xx} + 2u_{xy} + u_{yy} = 2 \qquad (1)$

Here $A = 1, B = 1, C = 1$, and so the discriminant $B^2 - AC = 1 - 1 = 0$, which means the equation is parabolic. Thus it gives one family of characteristic curve with root as $-\frac{B}{A}$:

$$\frac{dy}{dx} = -\frac{r_x}{r_y} = \frac{B}{A} = 1 \quad \Rightarrow \quad dy = dx$$

which is integrated to get:

$$y = x + c_1 \quad \text{or} \quad y - x = c_1$$

Identifying the constant c_1 with the canonical coordinate r, we have:

$$r = y - x$$

And since $y + x$ and $y - x$ are linearly independent, we select the second canonical coordinates as:

$$s = y + x$$

So that

$$\begin{vmatrix} y & -x \\ y & x \end{vmatrix} \neq 0 \quad \text{i.e.,} \qquad \text{non} - \text{singular}$$

Thus, evaluating the first and second derivatives of r and s with respect to x and y, we obtain:

$$r_x = -1, \qquad r_y = 1, \qquad s_x = 1, \qquad s_y = 1, \quad and \quad r_{xx} = r_{xy} = r_{yy} = s_{xx} = s_{xy} = s_{yy} = 0$$

Now substituting for u_{xx}, u_{xy} and u_{yy} in the equation, by applying (2.3), (2.4) and (2.5), we have:

$$u_{xx} = u_{rr}{r_x}^2 + 2u_{rs}r_xs_x + u_{ss}{s_x}^2 = u_{rr} - 2u_{rs} + u_{ss} \qquad (2)$$

$$u_{yy} = u_{rr}{r_y}^2 + 2u_{rs}r_ys_y + u_{ss}{s_y}^2 = u_{rr} + 2u_{rs} + u_{ss} \qquad (3)$$

$$u_{xy} = u_{rr}r_xr_y + 2u_{rs}(r_xs_y + r_ys_{x)} + u_{ss}s_xs_y = -u_{rr} + 2u_{rs}(-1+1) + u_{ss} = -u_{rr} + u_{ss} \qquad (4)$$

Substituting into equation (1) and evaluating, we have:

$$u_{rr} - 2u_{rs} + u_{ss} + 2(-u_{rr} + u_{ss}) + u_{rr} + 2u_{rs} + u_{ss} = 2$$

or

$$4u_{ss} = 2 \qquad \Rightarrow \qquad u_{ss} = \frac{1}{2} \qquad (5)$$

which is the required canonical form. Integrating (5) twice with respect to s, gives:

$$u_s = \frac{1}{2}s + f(r)$$

and

$$u(x, y) = \frac{1}{4}s^2 + sf(r) + g(r)$$

Now, substituting for r and s in this equation, we obtain the general solution:

$$u(x, y) = \frac{1}{4}(y + x)^2 + (y + x)f(y - x) + g(y - x)$$

where f and g are arbitrary functions of their arguments.

3. Equations of Mathematical Physics

The most frequently encountered PDE's in practice are members of the classical equations of mathematical physics. The majority of these can be obtained by suitably specialising the form:

$$\nabla^2 u + \alpha u = \beta u_{tt} + \gamma u_t - F \qquad (3.1)$$

where $\alpha, \beta,$ and γ are certain specified physical constants and F is of a special function of position (and probably time). The operator ∇^2 the Laplacian operator, and the quantity ∇^2 is called simply the Laplacian.

The Laplacian is a measure of the difference between the value of u at a point and the average value of u in a small neighbourhood of the point. Since this difference influences the further space-time evolution of the unknown function u in problems of diffusion processes, wave propagation, and potential theory, we find that the Laplacian is fundamental to most of the equations of mathematical physics. In rectangular coordinates the Laplacian takes the form:

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} \qquad (3.2)$$

It is often necessary to consider coordinate systems other than rectangular – the most advantageous in a particular problem generally being dictated by the shape of the region of interest. For such problems, we must find expressions comparable to (3.2) for the Laplacian in these other coordinate systems.

Specific equations arising out of (3.1) include the following:

$$\nabla^2 u = a^{-2} u_t \qquad \text{(heat equation)}$$

$$\nabla^2 u = c^{-2} u_{tt} \qquad \text{(wave equation)}$$

$$\nabla^2 u = 0 \qquad \text{(potential equation)}$$

$$\nabla^2 u + k^2 u = 0 \qquad \text{(Helmholtz equation)}$$

$$\nabla^2 u = -F \qquad \text{(Poisson equation)}$$

These PDE's play an important role in many diverse areas of application. However, other PDE's occur in certain applications that are not specialisations of (3.1), but will not be given separate treatment.

Example 3.1: Solve the 1-D wave equation problem: $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < L, t > 0$
with the boundary condition:
$u(x, 0) = f(x)$ for $0 \le x \le L$
$u_t(x, 0) = g(x)$ for $0 \le x \le L$

Solution:
Here $A = 1,\ B = 0,\ C = -c^2$. The discriminant $B^2 - AC = c^2 > 0$, and so the equation is hyperbolic. The ODE's to solve to obtain the two families of characteristic are:

$$\frac{dx}{dt} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \pm c$$

Which are integrated to get $x + ct = c_1,\ x - ct = c_2$. We identify the canonical coordinates r and s with the constants c_1 and c_2 to obtain:

$$r = x + ct, \quad s = x - ct \qquad (1)$$

Thus,

$$r_x = s_x = 1,\ r_t = c, \qquad s_t = -c, \qquad r_{xx} = r_{tt} = s_{xx} = s_{tt} = 0$$

and so,

$$u_{xx} = u_{rr} {r_x}^2 + 2u_{rs} r_x s_x + u_{ss} {s_x}^2 = u_{rr} + 2u_{rs} + u_{ss}$$
$$u_{tt} = u_{rr} {r_t}^2 + 2u_{rs} r_t s_t + u_{ss} {s_t}^2 = c^2 u_{rr} - c^2 u_{rs} + c^2 u_{tt} = c^2(u_{rr} - u_{rs} + u_{tt})$$

These are employed in $u_{tt} - c^2 u_{xx} = 0$ to get:

$$-4c^2 u_{rs} = 0, \quad or \quad u_{rs} = 0 \qquad (2)$$

Which is the canonical form of the PDE. Integrating (2) with respect to s yields:

$$u_r = p(r) \text{ and } u = p(r) + q(s)$$

Using (1), we obtain the general solution:

$$u(x,t) = p(x+ct) + q(x-ct) \qquad (3)$$

where p and q are arbitrary functions of their argument. (3) is called D'Alembert's solution of the 1-D wave equation: $u_{tt} - c^2 u_{xx} = 0$.
Therefore,

$$u_t(x,t) = cp'(x+ct) - cq'(x-ct) \qquad (4)$$

From (3) and (4), the initial conditions: $u(x,0) = f(x),\ \ u_t(x,0) = g(x)$ for $0 \le x \le L$ we have:

$$p(x) + q(x) = f(x) \quad \text{for } 0 \le x \le L \qquad (5)$$

$$cp'(x) + cq'(x) = f(x) \quad \text{for } 0 \le x \le L \qquad (6)$$

Multiplying the derivative of (5) by c and adding to (6) yields:

$$p'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x) \ \text{ for } \ 0 \le x \le L$$

which is integrated with respect to x to obtain:

$$p(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\tau)d\tau + K \ \text{ for } 0 \le x \le L \qquad (7)$$

where K is an integration constant. Employing (7) in (5) gives:

$$q(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\tau)d\tau - K \ \text{ for } 0 \le x \le L \qquad (8)$$

Replacing x with $x+ct$ in (7), and x with $x-ct$ in (8), and substituting these results in (3) yields:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_0^{x+ct} g(\tau)d\tau - \frac{1}{2c}\int_0^{x-ct} g(\tau)d\tau$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

provided $0 \le x \pm ct \le L$.

<u>Assignment</u>

Classify the PDE as hyperbolic, parabolic, or elliptic and find its general solutions:

1. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
2. $u_{xx} + a^2 u_{yy} = 0,\ (a \ne 0)$
3. $4u_{tt} - 12u_{xt} + 9u_{xx} = 0$
4. $u_{xx} + 2u_{xy} + 5u_{yy} = 0$

4. Separation of Variables

For a linear homogeneous PDE, it is sometimes possible to find a particular solution in the form of a product:

$$u(x,y) = X(x)Y(y)$$

The use of the above product, called the method of separation of variables may enable us to reduce a PDE to at least two ODEs. This general method of attack is mostly useful in solving PDEs of the type (3.1) above and may also be referred to as Bernoulli product method.

Theorem: (Superimposition Principle)

If $u_1, u_2, \dots, u_k$ are solutions of a homogeneous PDE, then the linear combination:

$$u = \sum_{n=1}^{k} c_n u_n$$

is also a solution, where $c_1, c_2, \dots, c_k$ are constants. Generally, if we have an infinite set of solutions of a linear homogeneous PDE denoted by $u_1, u_2, \dots$, we can construct another solution u by forming the infinite series:

$$u = \sum_{n=1}^{\infty} u_n$$

The combination of the separation of variables and the superimposition of solutions, sometimes called Fourier method will use Fourier cosine or sine series.

Example 4.1: Solve the heat conduction problem: $ku_{xx} = u_t$, for $k > 0,\ 0 < x < L,\ t > 0$

Boundary conditions

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

where k is a constant called the thermal diffusivity.

Solution: Employing the method of separation of variables, we assume a solution in the form:

$$u(x,t) = X(x)T(t) \qquad (1)$$

Substituting into the PDE, we have:

$$kX''T = XT' \ or \ \frac{X''}{X} = \frac{T'}{kT} \qquad (2)$$

Since the LHS of (2) is a function only of x and the RHS is a function only of t, it follows that both sides must be independent of both x and t, and so must be equal to a constant, say $-\lambda^2$. Thus,

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda^2 \qquad (3)$$

and hence,

$$X'' + \lambda^2 X = 0, \qquad X(0) = X(L) = 0 \qquad (4)$$

and

$$T' + k\lambda^2 T = 0 \qquad (5)$$

Since the boundary conditions $u(0,t) = u(L,t) = 0$, implies that $X(0)T(t) = X(L)T(t) = 0$.
Thus, the general solution of the ODE in (4) is:

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

where c_1 and c_2 are arbitrary constants. The boundary conditions in (4) gives:

$$X(0) = c_1 = 0$$

and

$$X(L) = c_1 \cos \lambda L + c_2 \sin \lambda L = c_2 \sin \lambda L = 0$$

Since $c_2 \neq 0$ for non-trivial solution, therefore $\sin \lambda L = 0$, which implies that:

$$\lambda L = n\pi, \ \ n = 1,2,\dots \quad or \ \ \lambda = \frac{n\pi}{L}, \ \ n = 1,2,\dots$$

The values of λ are called the eigenvalues of the problem. Therefore, the corresponding eigenfunctions are:

$$X_n(x) = c_{2n} \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2, \ldots \tag{6}$$

However, the general solution of (5) is:

$$T(t) = c_3 e^{-k\lambda^2 t}$$

where c_3 is an arbitrary constant.
Since $\lambda = \frac{n\pi}{L}$, we obtain:

$$T_n(t) = c_3 e^{\frac{-kn^2\pi^2 t}{L^2}}, \qquad n = 1,2, \ldots \tag{7}$$

Employing (6) and (7) in (1) yields:

$$u_n(x,t) = X_n(x)T_n(t) = A_n e^{\frac{-kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

where $A_n = c_{2n}c_{3n}$. By superimposition principle, we have:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{\frac{-kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \tag{8}$$

Applying the initial condition, $u(x,0) = f(x)$ gives:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

which is a Fourier sine series for $f(x)$. Thus,

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \;\; n = 1,2, \ldots \tag{9}$$

Hence the required particular solution of the problem is (8), where A_n is given by (9).

Example 4.2: Solve the Laplace equation: $u_{xx} + u_{yy} = 0$ for $0 < x < a, 0 < y < b$, subject to the boundary conditions:
$u(x,0) = u_x(0,y) = u_x(a,y) = 0$
$u(x,b) = f(x)$

Solution:
Let

$$u(x,y) = X(x)Y(y) \tag{1}$$

Then the Laplace equation becomes:

$$X''Y + Y''X = 0, \quad \text{or} \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

Therefore,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 (\text{constant})$$

and so,

$$X'' + \lambda^2 X = 0, \qquad X'(0) = X'(a) = 0 \tag{2}$$

and

$$Y'' - \lambda^2 Y = 0, \qquad Y(0) = 0, \qquad Y(b) = f(x) \tag{3}$$

Thus, the general solution of the ODE in (2) is:

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \tag{4}$$

where c_1 and c_2 are arbitrary constants.
Thus,

$$X'(x) = -\lambda c_1 \sin \lambda x + \lambda c_2 \cos \lambda x \tag{5}$$

Applying the boundary condition, we have:

$$X'(0) = -\lambda c_1 \sin(0) + \lambda c_2 \cos(0) = \lambda c_2 = 0$$

But $\lambda \neq 0$, so $c_2 = 0$, which gives: $X(x) = c_1 \cos \lambda x$

Next, applying the second BC, we have:

$$X'(a) = -\lambda c_1 \sin \lambda a + \lambda c_2 \cos \lambda a = 0$$

But since $c_2 = 0$ and $c_1 \neq 0$, we have:

$$-\lambda c_1 \sin \lambda a = 0 \quad \Rightarrow \lambda = 0 \text{ or } \sin \lambda a = 0 \quad \Rightarrow \lambda a = n\pi, \quad n = 1,2, \dots$$

and so for $\lambda \neq 0$, the eigenvalues of the problem are: $\lambda = \frac{n\pi}{a}, n = 1,2, \dots$

Thus, the corresponding eigenfunction are:

$X(x) = c_1 \cos \lambda x$ and $X_n(x) = c_1 \cos \frac{n\pi}{a} x,\ n = 1,2, \dots$ (6)

Next, the general solution of the ODE (3) is:

$$Y(y) = c_3 \cosh \lambda y + c_4 \sinh \lambda y \qquad (7)$$

Where c_3 and c_4 are arbitrary constants.

Applying the BC for (7), $Y(0) = 0$, we have:

$$Y(0) = c_3 \cosh(0) + c_4 \sinh(0) = c_3 = 0$$

Therefore, for $\lambda \neq 0$, we have:

$$Y(y) = c_4 \sinh \lambda y \quad \text{and} \quad Y_n(y) = \sinh \frac{n\pi}{a} y, \quad n = 1,2, \dots \qquad (8)$$

However, if $\lambda = 0$, then (3) becomes:

$$Y'' = 0 \qquad (9)$$

which yields:

$$Y(y) = c_5 y + c_6 \qquad (10)$$

where c_5 and c_6 are arbitrary constants.

Thus,

$$Y(0) = c_5(0) + c_6 = 0, \ \text{i.e.}, c_6 = 0$$

So that from (10):

$$Y(y) = c_5 y \qquad (11)$$

Now, employing (6), (8) and (11) in (1) and utilizing superimposition principle, we obtain:

$$u(x, y) = \frac{A_0}{1} y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a} \qquad (12)$$

Which is a Fourier cosine series for $f(x)$ and $c_5 = A_0,\ c_{1n} c_{4n} = A_n$

Thus, applying the final BC, $Y(b) = f(x)$, we have:

$$A_0 b = f(x) \quad \text{or} \quad A_0 b = \frac{1}{a} \int_0^a f(x) dx \qquad (13)$$

and

$$A_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a} dx, \quad n = 1,2, \dots \qquad (14)$$

Therefore,

$$A_0 = \frac{1}{ab} \int_0^a f(x) dx \qquad (15)$$

and

$$A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi x}{a} dx, \quad n = 1,2, \dots \qquad (16)$$

Hence the required particular solution is (12), with A_0 and A_n given by (15) and (16) respectively.

Example 4.3: Use the separation of variables to solve: $u_x + 2u_y = 0, \quad u(0,y) = 3e^{-2y}$.

Solution: By writing $u(x,y) = X(x)Y(y)$ and substituting this product from into the PDE, we obtain:

$$X'(x)Y(y) + 2X(x)Y'(y) = 0$$

which can also be expressed in the form:

$$\frac{X'}{2X} = -\frac{Y'}{Y}$$

Here we have separated the variables, and by equating each side of the equation to the constant $-\lambda^2$, we get the ODEs:

$$X' + 2\lambda^2 X = 0 \qquad (1)$$

and

$$Y' - \lambda^2 Y = 0 \qquad (2)$$

Equations (1) and (2) have solutions given respectively by:

$$X(x) = c_1 e^{-2\lambda^2 x} \qquad \text{and} \qquad Y(y) = c_2 e^{\lambda^2 y}$$

and thus,

$$u(x,y) = X(x)Y(y) = c_1 e^{-2\lambda^2 x} \cdot c_2 e^{\lambda^2 y} = Ae^{-\lambda^2(2x-y)}$$

where $A = c_1 c_2$.

If we apply the auxiliary condition, we find:

$$u(0,y) = Ae^{\lambda^2 y} = 3e^{-2y}$$

and hence, $A = 3, \ \lambda^2 = -2$.

Our solution is therefore:

$$u(x,y) = 3e^{2(2x-y)}$$

Assignment

Find the solutions by the method of separation of variables:

1. $u_x = u_y, \quad u(0,y) = 2e^{3y}$.
2. $u_x + u = u_y, \ u(x,0) = 4e^{-3x}$
3. $u_{xx} = u_{tt}, \quad u(0,t) = 0, \ u(\pi,t) = 0, \ u(x,0) = \sin 3x, \ \ u_t(x,0) = 0$.
4. $x^2 u_{xy} + 3y^2 u = 0, \quad u(x,0) = e^{\frac{1}{x}}$
5. $u_{xx} = \frac{2}{k} u_t + u, \ u(1,t) = 0, \ u_x(0,t) = -be^{-kt}, \ (k, b$ constants).

5. Solutions by ODE Methods

Most PDEs must be solved by a general solution technique, such as separation of variables or a transform method, or in some cases by a numerical procedure. However, occasionally, the PDE of interest is simple enough that its form may suggest a method of solution.

Example 5.1: Find a solution of the boundary value problem: $u_{xy} - u_y = 5;$

$$u_y(0,y) = 3y^2, \ u(x,0) = 0$$

Solution: By writing the PDE as:

$$\frac{\partial}{\partial y}(u_x - u) = 5$$

we can hold x fixed and integrate with respect to y to obtain:

$$u_x - u = 5y + F(x)$$

where F is an arbitrary, differentiable function of x. For fixed y, this last equation is a first-order liner DE, whose general solution is readily found to be:

$$u(x,y) = e^x G(y) - 5y + e^x \int e^{-x} F(x)dx = e^x G(y) - 5y + H(x)$$

where we define $H(x) = e^x \int e^{-x} F(x)dx$.
Now, imposing the first auxiliary condition, we have:

$$u_y(0,y) = G'(y) - 5 = 3y^2$$

from which we deduce;

$$G(y) = y^3 + 5y + c$$

where c is a constant. Hence:

$$u(x,y) = e^x(y^3 + 5y + c) - 5y + H(x)$$

and second condition leads to:

$$u(x,0) = ce^x + H(x) = 0 \quad \text{or} \quad H(x) = -ce^x$$

Our solution now takes the form:

$$u(x,y) = e^x(y^3 + 5y) - 5y$$

Assignment

Use ODE methods to find the solution satisfying the prescribed boundary conditions:

1. $u_x = \sin y\,;\quad u(0,y) = 0$
2. $u_{yy} = x^2 \cos y\,;\quad u(x,0) = 0,\ \ u\left(x, \frac{\pi}{2}\right) = 0$
3. $u_{xy} = 4xy + e^x;\quad u_y(0,y) = y,\ \ u(x,0) = 2$
4. $u_{xy} + 4u_x = 2x;\quad u(0,y) = 1,\ \ u_x(x,0) = 0$
5. $u_{xy} = u_x + 2;\quad u(0,y) = 0,\ \ u_x(x,0) = x^2$

6. Fourier Series

One of the most important problems in mathematical analysis is the determination of various representations of a given function. A particular representation of a function often enables us to deduce properties of that function that are not as readily ascertained by a different representation. Power series are especially useful in this regard, but here we wish to extend our notion of infinite series to include those involving sines and cosines, called Fourier series. When the function involved is either even or odd, the full Fourier trigonometric series reduces to either a cosine series for an even function or a sine series for an odd function.

Fourier analysis, or harmonic analysis as it is now often called, has turned out to be tremendously important in virtually all areas of pure and applied mathematics and the physical sciences. It is one of the best examples of a mathematics tool that was invented to solve a specific problem and has turned out to be an important tool for solving many other problems.

6.1 Fourier Series of Periodic Functions

A function f is called periodic if there exists a constant $T > 0$ for which

$$f(x+T) = f(x) \text{ for all } x$$

The smallest value of T for which the property holds is called the fundamental period, or simply, the period (see fig. 6.1). It follows that if:

$$f(x+T) = f(x)$$

Then also;

$$f(x \pm T) = f(x \pm 2T) = f(x \pm 3T) = \cdots = f(x)$$

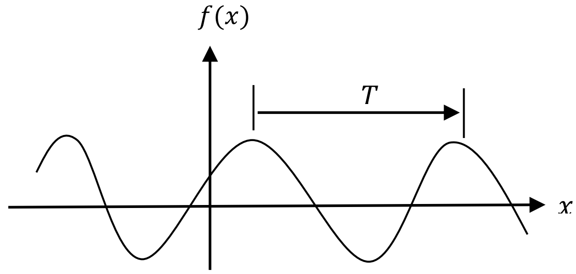

Fig. 6.1

Periodic functions appear in a wide variety of physical problems, such as those concerning vibrating springs and membranes, planetary motion, a swinging pendulum, and musical sounds, etc. Many of these Phenomena involve periodic function of a complicated nature, so in order to better understand such functions it is desirable to express them in terms of a set of simple periodic functions. Doing so has the effect of decomposing a periodic phenomenon into its simple harmonic components.

Perhaps the simplest set of periodic functions is given by:

$$1; \ \cos x, \sin x; \ \cos 2x, \sin x; \ \cos 3x, \sin 3x; \ \ldots;$$

which all have period 2π, it may seem reasonable to look for a representation of f in terms of the above simple sinusoidal functions. The series arising in this connection will be of the form:

$$f(x) = A_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (6.1)$$

Where $A_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, are constants. Such a series is called a Fourier (trigonometric) series.

Our method depends upon the evaluation of certain definite integrals involving the sines and cosines appearing in (6.1). First, if n and k are any nonzero integers, it can be shown that:

$$\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = \int_{-\pi}^{\pi} \sin nx \cos kx \, dx = 0 \qquad (6.2)$$

and also,

$$\int_{-\pi}^{\pi} \cos nx \cos kx \, dx = \int_{-\pi}^{\pi} \sin nx \sin kx \, dx = \begin{cases} 0, & k \neq n \\ \pi & k = n \end{cases} \qquad (6.3)$$

These integral formulas can be derived directly through simple integration techniques. The integral relations (6.2) suggest that integrating (6.1) from $-\pi$ to π will greatly simplify the right–hand side. To do this, we must tacitly assume that termwise integration is justified. Proceeding in this fashion, we obtain:

$$\int_{-\pi}^{\pi} f(x) \, dx = A_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right)$$

In view of the integral relations (6.2), we see that each term of the series integrates to zero, and from the remaining nonzero integrals, we find;

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \qquad (6.4)$$

This identifies the constant A_0 as the average value of $f(x)$ over the interval $[-\pi,\ \pi]$. Next, we multiply (6.1) by $\cos kx$ and integrate termwise to obtain:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = A_0 \int_{-\pi}^{\pi} \cos kx\ \ dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos kx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\ \cos kx\,dx\right)$$

Because (6.2) and (6.3), all terms integrate to zero except for the coefficient of a_n corresponding to $n = k$, and here we get:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k \int_{-\pi}^{\pi} \cos^2 kx\,dx = \pi a_k$$

or

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad k = 1,2,3,\ldots \qquad (6.5)$$

In the same fashion, if we multiply the series (6.1) by $\sin kx$ and integrate the result termwise, we generate the final formula:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad k = 1,2,3,\ldots \qquad (6.6)$$

Constants defined by (6.4-6.6) are known as Fourier coefficients (also called Euler's formulas). It is customary in the literature to set:

$$A_0 = \frac{a_0}{2}$$

so that we can write the above formula more compactly as:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 1,2,3,\ldots \qquad (6.7)$$

(now changing the index back to n) and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1,2,3,\ldots \qquad (6.8)$$

Thus, (6.1) now takes the form:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (6.9)$$

Writing the constant term as $\frac{a_0}{2}$ does not aid in its computation, only in the compactness of the formula (6.7). In general, we find that a_0 must be evaluated separately from the rest of the a's.

Example 6.1: Find the Fourier series of the function:

$$f(x) = |x|, -\pi \le x \le \pi, \qquad f(x + 2\pi) = f(x)$$

Solution:

The substitution of $f(x) = |x|$ into (6.7) and (8) leads to:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \pi$$

$$a_n = -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \frac{2}{\pi n^2}[cox\,n\pi - 1], \quad n = 1,2,3,\ldots$$

and

$$b_n = -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx = 0, \quad n = 1,2,3,\ldots$$

Since $\cos n\pi = (-1)^n$, $n = 1,2,3,\ldots$, we can write:

$$a_n = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} -\frac{4}{\pi n^2}, & n = 1,3,5,\ldots \\ 0, & n = 2,4,6,\ldots \end{cases}$$

and thus the Fourier series becomes:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{\cos nx}{n^2}$$

By replacing the index n with new index $(2n-1)$, we can also express this Fourier series in the form:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Example 6.2: Find the Fourier series of the periodic function f that is defined by:

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi, \end{cases} \qquad f(x + 2\pi) = f(x).$$

Solution:
The Fourier coefficients are given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \begin{cases} -\frac{2}{\pi n^2}, & n = 1,3,5,\ldots \\ 0, & n = 2,4,6,\ldots \end{cases}$$

and

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx = \frac{(-1)^{n-1}}{n}, \quad n = 1,2,3,\ldots$$

Substituting these results into the into the series (6.9), we obtain:

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$$

or

$$f(x) = \frac{\pi}{4} - \sum_{n=1}^{\infty}\left[\frac{2}{\pi} * \frac{\cos(2n-1)}{(2n-1)^2} + \frac{(-1)^n}{n}\sin nx\right]$$